Identification of hyperbolic discount factor in dynamic discrete choice model with multiple terminating actions

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Abstract

This paper studies identification of quasi-hyperbolic discount dynamic discrete choice models in both finite and infinite horizons, exploring the unique features of the presence of multiple terminating actions. Under economically meaningful exclusion restrictions, the identification of discount factors is characterized by polynomial moment conditions. The presence of multiple terminating actions greatly reduces the complication of the identification and also helps relax the restrictions imposed on the flow utility function. This paper also examines the impact of estimating the 'underlying' hyperbolic discounting model as the prevalent exponential discount model. I find that such misspecification could lead to misleading policy implications.
Keywords: identification, hyperbolic discount factor, dynamic discrete choice model

1 Introduction

The time inconsistency has got extensive attention from economists. The prevalent approach to model time inconsistency is (quasi-)hyperbolic discounting. Such a framework is convenient in studying problems like addiction and self-control. (*e.g.* O'Donoghue and Rabin, 1999, [1]). In traditional economics, agents are assumed to discount the future value in the exponential rate, which is similar to the compound interest rate. However, there are more and more evidence from experiments showing that people may not only evaluate the future in this way. In the lab experiments, considerable amount of people tended to show present-biased preference. For example, when people are asked about "whether you like 1 dollar tomorrow or 2 dollars two days later", most people would choose 1 dollar tomorrow. However, when asked about "whether you like 1 dollar one month later, or 2 dollars one month plus one day later", some people would choose 2 dollar case. In the present-biased model, the intertemperal optimization problem can be regarded as consisting of many independent selves, one in each period. Every self has different time preference, and they are inconsistent with each other. That is, the choices made by current self today might not be preferred by the future self, even if they know the same information. This kind of inconsistency generates the commitment and self-control problem, and also leaves room for policy intervention.

There are two types of present-biased people called naive and sophisticated agents. The naive agents are those who care about the future in present-biased mode but perceive that they will discount the future as if they are using exponential rate tomorrow. That is, naive agents are present-biased in current period, but think that they will not be biased in the future (when tomorrow really comes, they are still present-biased). The sophisticated agents are present-biased in current period, and they know that they will be present-biased in the future.

One big problem in adopting the hyperbolic discounting factor model is joint identification of both present-biased discounting factor and the exponential discount factor. In the standard dynamic discrete choice (DDC) model, the discount factor is non-parameterically underidentified. Fang and Wang (2015) [2] provide the identification for partially naive case. Abbring and Dalijord (2020) [3] shows that the proof in Fang and Wang (2015) [2] is incorrect. Abbring, Daljord and Iskhakov (2019) [4] also gives the set identification result of discount factor for sophisticated agents in finite horizon case. Mahajan, Michel and Tarozzi (2020) [5] gives the identification result of present-biased time preference with finite horizon. Chan (2017) [6] estimates a DDC model with hyperbolic discounting factor to analyze the welfare dependence with the identification argument of Fang and Wang (2015) [2].

Most research above focuses on the finite horizon problem. One big disadvantage of using finite horizon problem is that it is very easy to characterize the optimal behaviors via backward induction. However, the identification in this circumstance usually requires getting access to the data in the final period. This data requirement sometimes is restrictive. On the other hand, there are plenty of cases when the agents are assumed to make decision across a long time, such as house mortgage, saving in pension. In those cases it would be hard to observe the final period data. When the decision is over a long time span, one could expect that the behaviors further away from the terminal period might be similar to that of the infinite horizon framework. Moreover, in practice, there are plenty examples that agents do not have a clear recognition of the terminal period, i.e., random stopping problems. In circumstances like this, a infinite horizon model might be more appropriate. Due to its difficulty in identification, infinite horizon DDC with hyperbolic discounting factor has not been sufficiently explored.

In this paper, I provide the conditions of identification of hyperbolic discounting factor in DDC model, in both finite horizon and infinite horizon scenarios. The identification extends the existing results, i.e., Rust (1987) [7], Hotz and Miller (1993) [8]. The contribution of this paper is to explore the unique features of multiple terminating actions, motivated by the borrower's decisions after taking up a reverse mortgage in the reality, i.e., Blevins et al (2020) [9]. Terminating actions are those actions that will terminate the model after being adopted. The introduction of multiple terminating action can help to reduce the normalization in flow utility function for finite horizon exponential case [9]. Taking housing mortgage as an example, in each period, the agents can choose to make interest payment and continue the mortgage, but they could also choose to default or refinance the mortgage, which will lead to the termination of the mortgage contract. In this case, there are two kinds of terminating actions, default and refinance. Then housing mortgage can be regarded as model with multiple terminating actions. I show that multiple terminating action can also help reduce the normalization requirement in infinite horizon hyperbolic discounting case. With normalizing the reference flow utility for at least two states, present-biased discounting factor, exponential discounting factor and flow utility can be identified.

This paper also contributes to the existing literature in that I use Monte Carlo simulations to understand the difference between the hyperbolic discounting model and the exponential discounting model. Specifically, I first generate the agent's action according to the optimal behaviors in the hyperbolic discounting model. I then estimate the model primitives for both the hyperbolic discounting model and the exponential discounting model. I find that exponential discounting model tends to underestimate the exponential discounting factor, especially when the optimal present-biased discounting factor is small. In this case, misspecification could also estimate the flow utility wrongly. I then simulate the agent's actions under different policy changes using both estimates to understanding the risk of mis-specification of a hyperbolic discounting model as an exponential discounting model. [1] I find that mis-specification could result in misleading counterfactual outcomes.

The rest of the paper is organized in the following structure: section 2 provides the DDC model setup in both infinite horizon case and finite horizon case; section 3 discusses the conditions in identification of discounting factors; section 4 provides simulation result in a simple case to show the effect of the misspecification; section 5 concludes.

[1] There are research of normalization in reference utility impacting the counterfactual analysis, *e.g.* Norets and Tang (2014) [10], Aguirregabiria and Suzuki (2014) [11], Kalouptsidi et al (2015) [12].

2 A General Model of hyperbolic discounting

Consider a dynamic discrete choice model that an individual needs to decides her action in period $t = 1, ..., T$, where T can be finite or infinite. Let u_t represent the individual's utility at period t. The life-time utility for an individual with a present-biased time preference at period t can be represented as

$$U_t(u_t, u_{t+1}, ...u_T) \equiv u_t + \beta\delta \sum_{t'=t+1}^{T} \delta^{t'-t-1} u_k, \tag{1}$$

where β captures the individual's present-biased time preference, and δ is the long-term discounting rate, which is the same as the discount rate in the traditional exponential discounting framework. This hyperbolic discount factor framework nests the exponential framework that is equivalent to $\beta = 1$, meaning that the agent does not have present-biased time preference (Abbring, Daljord and Iskhakov (2019) [4]).

The dynamic process can be summarized as the follows. In each period t, the agent chooses an action d_t from a choice set $\mathcal{D} = \{1, 2, ..., K\}$. Prior to making the choice, the agent can observe state variables x_t and $\epsilon_t \in \{\epsilon_{1,t}, ..., \epsilon_{K,t}\}$, where state variables x_t are observable to both the agent and the econometrician, while state variables ϵ_t are only observable to the agent.

Following the existing literature, we model that the action-specific private information entering the agent's utility function additively separable. That is, the utility function in period t can be represented as:

$$u_t(x, k, k_{-1}, \epsilon) = u_{k,t}(x, k_{-1}) + \epsilon_{k,t}, \tag{2}$$

where k_{-1} is the action in the last period. We allow the lagged action entering the utility function for flexibility. In some applications, the lagged action does not affect the flow utility. The state variables x_t are assumed to have finite support $\mathcal{X} = \{x_1, x_2, \cdots, x_J\}$ and have stationary Markov process controlled by d_t. The transition matrix is defined as $Q_k(x'|x)$ if $k \in \mathcal{D}$ is chosen. The shocks $\epsilon_{k,t}$ is independent from x_t, prior states and choices. The shocks are also independent over time, across choices, and have type 1 extreme value distributions for analytical simplification.

The agent then chooses her action every period t to maximize the life-time utility at time t by taking into account the future utility generated by the future selfs who is also a utility maximizer. For those present-biased agents, there are two types of them depending on how the current self views their future selfs: sophisticated and naive. The sophisticated agent knows that her future self is also present-biased and maximizes the life time utility characterized by Equation 1, while the naive agent thinks that the future self is time-consistent and maximizes the life-time utility characterized by Equation 1 with $\beta = 1$. Let $\sigma_t : \mathcal{X} \times \mathbb{R}^K \to \mathcal{D}$ be an arbitrary choice strategy in period t and $\boldsymbol{\sigma}_t = \{\sigma_\tau\}_{\tau=t}^{\infty}$ be the life-time strategy profile. In what follows, we are going to characterize the agent's optimal decision for both sophisticated and naive agent separately.

An intuitive example to understand sophisticated agent's decision process is: a smoker is considering quitting smoker every period, a sophisticated agent is aware that it is difficult to quit smoking now, and it is difficult to quit smoking in the future, but they know that from the long-term perspective they should quit it. There is conflict between the current self and the self that focuses on the long term benefit; and the agent is aware of this conflict. On the other hand, there is not conflict between the *perceived* future self and the future self.

An intuitive example to understand naive agent's decision process is: a smoker is considering quitting smoker every period. The smoker thinks that she will quit tomorrow because the discount rate of future benefit from quitting smoking tomorrow is only δ while it is $\delta\beta$ today. However, when tomorrow comes, the smoker chooses to continue smoking and thinks that she will quit the next day. Naive agent tends to postpone unpleasant action and thinks that she will do it in the future but she continue postpone it when tomorrow comes. There is conflict between the *perceived* future self and the future self. Specifically, the *perceived* future self is time-consistent and maximizes the long term benefit while the actual decision future self makes is bounded by the present-biased preference.

To incorporate the prevalent feature of a dynamic discrete choice of terminal actions, we introduce a dummy variable $I_{k,k_{-1}}$ to control whether the decision process enters into the next period or not. Specifically, $I_d = 0$ indicates the dynamic process ends while $I_{k,k_{-1}} = 1$ means the dynamic process continues.

2.1 Infinite Horizon Framework

We might follow the existing literature and introduce a factor $\tilde{\beta}$ to measure the level of naiveness so that we can model the model in a unified framework.

Infinite-horizon Sophisticated agent optimization In infinite horizon model, the flow utility is stationary and thus can be described as $u_k(x, k_{-1})$. Also, the agent's *current choice-specific value function* is stationary and can be generally described as

$$w_k(x, k_{-1}; \tilde{\boldsymbol{\sigma}}) = u_k(x, k_{-1}) + I_{k,k_{-1}}\beta\delta \sum_{x'} v(x', k; \tilde{\boldsymbol{\sigma}}) Q_k(x'|x), \tag{3}$$

where $v(x', k)$ is the *perceived* long-run value function defined as

$$v(x, k_{-1}; \tilde{\boldsymbol{\sigma}}) = \mathbb{E}_{\epsilon'} V(x, k_{-1}, \epsilon; \tilde{\boldsymbol{\sigma}}) \equiv u_{\tilde{\sigma}}(x, k_{-1}) + \epsilon(\tilde{\sigma}) + \delta \mathbb{E}\left[V(x', \tilde{\sigma}, \epsilon'; \tilde{\boldsymbol{\sigma}})|x, \tilde{\sigma}\right], \tag{4}$$

where $V(x, k_{-1}, \epsilon; \tilde{\boldsymbol{\sigma}})$ is the stationary value function under the *perceived* future self's strategy profile $\tilde{\boldsymbol{\sigma}}$. A forward-looking agent faces the tradeoff between current utility and future values discounted disproportionally by factor $\beta\delta$, while all future utilities are discounted geometrically by factor δ.

Since the agent is sophisticated, her perceptions of the future strategies are consistent with her current strategy due to stationary. Thus, in equilibrium, her current selves use a *perception perfect strategy* (O'Donoghue and Rabin, 1999), which is a strategy profile $\boldsymbol{\sigma}^*$ such that each σ^* is a best response to her perceived future strategy profile $\boldsymbol{\sigma}^*$:

$$\sigma^*(x, k_{-1}, \epsilon) = \arg\max_{k\in\mathcal{D}}\{w_k(x, k_{-1}; \boldsymbol{\sigma}^*) + \epsilon_k\}. \tag{5}$$

Following the existing literature, we further define the conditional choice probability (CCP) that represents the probability each action being chosen given the current state variable. Thus, we can characterize the equilibrium CCPs via the following equation:

$$p_k(x, k_{-1}) \equiv Pr[\sigma^*(x, k_{-1}, \epsilon) = k] = Pr[w_k(x, k_{-1}; \boldsymbol{\sigma}^*) + \epsilon_k \geq w_{k'}(x, k_{-1}; \boldsymbol{\sigma}^*) + \epsilon_k, \quad \forall k' \neq k]$$

for all $k \in \mathcal{D}; x \in \mathcal{X}$. Consequently, we can characterize the agent's optimal decision via the equilibrium CCPs. Because we assume that the private shocks follow the type-1 extreme value distribution, the equilibrium CCPs can be represented as

$$p_k(x, k_{-1}) = \frac{w_k(x, k_{-1}; \boldsymbol{p})}{\sum_{k'} w_{k'}(x, k_{-1}; \boldsymbol{p})}, \tag{6}$$

where the perceived long-run value function $v(x, k_{-1}; \boldsymbol{p})$ can be further represented as

$$\begin{aligned} v(x, k_{-1}; \boldsymbol{p}) &= \mathbb{E}_\epsilon\left[u_{\sigma^*(k_{-1},x,\epsilon)}(x, k_{-1}) + \epsilon(\sigma^*) + \delta\sum_{x'} v(x', \sigma^*; \boldsymbol{\sigma}'^*)Q_{\sigma^*(x,\epsilon')}(x'|x)\right] \\ &= \mathbb{E}_\epsilon\left[w_{\sigma^*(k_{-1},x,\epsilon)}(x, k_{-1}) + \epsilon(\sigma^*) + \delta(1-\beta)\sum_{x'} v(x', \sigma^*; \boldsymbol{\sigma}'^*)Q_{\sigma^*(x,\epsilon')}(x'|x)\right] \\ &= \mathbb{E}_\epsilon max_k\left[w_k(x, k_{-1}) + \epsilon_k\right] + \mathbb{E}_\epsilon\left[\delta(1-\beta)\sum_{x'} v(x', \sigma^*; \boldsymbol{\sigma}'^*)Q_{\sigma^*(x,\epsilon)}(x'|x)\right] \\ &= \gamma - logp_K(x, k_{-1}) + w_K(x, k_{-1}) + \delta(1-\beta)\sum_k I_{k,k_{-1}}\sum_{x'} v(x', k; \boldsymbol{p})Q_k(x'|x)p_k(x, k_{-1}), \end{aligned} \tag{7}$$

which is a one-to-one mapping for the long-term value. Note that the long-term value is an aggregation over the choice-specific values associated with all possible actions being optimal. Consequently, the optimal behaviors of the agent can be characterized by Equations 3, 6, and 7. By some fixed point theorem, the perception perfect strategy exists and unique now

2.2 Finite Horizon Framework

For any kind of agent, the choice-specific value function at period T can be written as

$$w_{k_T,T}(x_T, k_{T-1}) = u_{k_T,T}(x_T, k_{T-1}), \quad k_T \in \mathcal{D} \tag{8}$$

For period $t \neq T$, since choosing default in last period will affect continuation of the loan, choice specific value function can be expressed as

$$w_{k_t,t}(x_t, k_{t-1} = 0) = \begin{cases} u_{0,t}(x_t, 0) + \beta\delta \sum\limits_{x_{t+1}\in\mathcal{X}} V_{t+1}(x_{t+1}, 0)Q_0(x_{t+1}|x_t), & k_t = 0 \\ u_{1,t}(x_t, 0) + \beta\delta \sum\limits_{x_{t+1}\in\mathcal{X}} V_{t+1}(x_{t+1}, 1)Q_1(x_{t+1}|x_t), & k_t = 1 \\ u_{2,t}(x_t, 0), & k_t = 2 \end{cases} \tag{9}$$

$$w_{k_t,t}(x_t, k_{t-1} = 1) = \begin{cases} u_{0,t}(x_t, 1) + \beta\delta \sum\limits_{x_{t+1}\in\mathcal{X}} V_{t+1}(x_{t+1}, 0)Q_0(x_{t+1}|x_t), & k_t = 0 \\ u_{1,t}(x_t, 1), & k_t = 1 \\ u_{2,t}(x_t, 1), & k_t = 2 \end{cases} \tag{10}$$

For consistent agent, the present-biased discount factor $\beta = 1$, while for inconsistent agent (no matter sophiticated and naive agent), the present-biased discount factor $\beta \neq 1$.

Given the form of the choice-specific value function and the assumption of type-I extreme value distribution of the random term, we can get the form of conditional choice probability (CCP):

$$P_{k_t,t}(x_t, k_{t-1}) = \frac{exp(w_{k_t,t}(x_t, k_{t-1}))}{\sum\limits_{j\in\mathcal{D}} exp(w_{j,t}(x_t, k_{t-1}))} \tag{11}$$

For the terminal period choice-specific value function it can be seen that CCPs are the same for all kinds of agents. That is,

$$P_{k_T,T}(x_T, k_{T-1}) = \frac{exp(u_{k_T,T}(x_T, k_{T-1}))}{\sum\limits_{j\in\mathcal{D}} exp(u_{j,T}(x_T, k_{T-1}))} \tag{12}$$

Then the perceived long-run value function for terminal period is (action 2 is the reference action)

$$\begin{aligned} V_T(x_T, k_{T-1}) &= \mathbb{E}_{\epsilon_T}[u_{\sigma_T(x_T,k_{T-1},\epsilon_T),T}(x_T, k_{T-1}) + \epsilon_T(\sigma_T)] \\ &= \mathbb{E}_{\epsilon_T}[u_{\sigma_T(x_T,k_{T-1},\epsilon_T),T}(x_T, k_{T-1}) - u_2(x_T) + \epsilon_T(\sigma_T)] + u_{2,T}(x_T, k_{T-1}) \\ &= \gamma - \ln P_{2,T}(x_T, k_{T-1}) + u_{2,T}(x_T, k_{T-1}) \end{aligned} \tag{13}$$

[2] Note that here $\sigma_T(x_T, k_{T-1}, \epsilon_T)$ is the best choice for period T. The last equality holds under the type-I extreme vale distribution of ϵ_T. Also note that for all kinds of agents, this perceived long-run value function for terminal period holds. The reason is that at the terminal period, only the flow payoff works, and then all agents can "perceive" the same terminal period value. Then plug $V_T(x_T, k_{T-1})$ in the expression of choice-specific value function in period $T-1$, and the CCP for period $T-1$ can be derived for all kinds of agents. Note that the only differences among all

[2] Since there is terminating action, the Euler's constant γ does have an impact in the perceived long-run value function here and it may not be normalized casually.

agents' CCP at period $T-1$ are the present-biased factor in 9 and 10. If we assume that β are the same for sophisticated agent and naive agent, then the solutions for two-period problem are exactly the same for them.

Then let's solving the three-period case for all kinds of agents. Similar to two-period case, the consistent agent will iterate for another period. Now the differnce comes from the perceived long-run value function for period $T-1$:

$$\begin{aligned}
V_{T-1}(x_{T-1}, k_{T-2}) &= \mathbb{E}_{\epsilon_{T-1}}[u_{\sigma_{T-1}(x_{T-1},k_{T-2},\epsilon_{T-1}),T-1}(x_{T-1}, k_{T-2}) + \epsilon_{T-1}(\sigma_{T-1}) \\
&+ \delta \sum_{x_T \in \mathcal{X}} V_T(x_T, \sigma_{T-1}) Q_{\sigma_{T-1}}(x_T|x_{T-1})] \\
&= \mathbb{E}_{\epsilon_T}[w_{\sigma_{T-1}(x_{T-1},k_{T-2},\epsilon_{T-1}),T-1}(x_{T-1}, k_{T-2}) + \epsilon_{T-1}(\sigma_{T-1})] \\
&= \gamma - \ln P_{2,T-1}(x_{T-1}, k_{T-2}) + w_{2,T}(x_{T-1}, k_{T-2})
\end{aligned}$$

This expression is different from period T in the continuation value. Then plug this into the expression of choice-specific value function in period $T-2$, and CCP for all periods can be characterized.

For the sophisticated agent, the perceived long-run value function for period $T-1$ is

$$\begin{aligned}
V_{T-1}(x_{T-1}, k_{T-2} = 0) &= \mathbb{E}_{\epsilon_{T-1}}[u_{\sigma_{T-1}(x_{T-1},0,\epsilon_{T-1}),T-1}(x_{T-1}, 0) + \epsilon_{T-1}(\sigma_{T-1}) \\
&+ \delta \sum_{x_T \in \mathcal{X}} V_T(x_T, \sigma_{T-1}) Q_{\sigma_{T-1}}(x_T|x_{T-1})] \\
&= \mathbb{E}_{\epsilon_T}[w_{\sigma_{T-1}(x_{T-1},0,\epsilon_{T-1}),T-1}(x_{T-1}, 0) + \epsilon_{T-1}(\sigma_{T-1}) \\
&+ \delta(1-\beta) \sum_{x_T \in \mathcal{X}} V_T(x_T, \sigma_{T-1}) Q_{\sigma_{T-1}}(x_T|x_{T-1})] \\
&= \gamma - \ln P_{2,T-1}(x_{T-1}, 0) + w_{2,T}(x_{T-1}, 0) \\
&+ \delta(1-\beta) P_{0,T-1}(x_{T-1}, 0) \sum_{x_T \in \mathcal{X}} V_T(x_T, 0) Q_0(x_T|x_{T-1}) \\
&+ \delta(1-\beta) P_{1,T-1}(x_{T-1}, 0) \sum_{x_T \in \mathcal{X}} V_T(x_T, 1) Q_1(x_T|x_{T-1})
\end{aligned}$$

$$\begin{aligned}
V_{T-1}(x_{T-1}, k_{T-2} = 1) &= \mathbb{E}_{\epsilon_{T-1}}[u_{\sigma_{T-1}(x_{T-1},1,\epsilon_{T-1}),T-1}(x_{T-1}, 1) + \epsilon_{T-1}(\sigma_{T-1}) \\
&+ \delta \sum_{x_T \in \mathcal{X}} V_T(x_T, \sigma_{T-1}) Q_{\sigma_{T-1}}(x_T|x_{T-1})] \\
&= \mathbb{E}_{\epsilon_T}[w_{\sigma_{T-1}(x_{T-1},1,\epsilon_{T-1}),T-1}(x_{T-1}, 1) + \epsilon_{T-1}(\sigma_{T-1}) \\
&+ \delta(1-\beta) \sum_{x_T \in \mathcal{X}} V_T(x_T, \sigma_{T-1}) Q_{\sigma_{T-1}}(x_T|x_{T-1})] \\
&= \gamma - \ln P_{2,T-1}(x_{T-1}, 1) + w_{2,T-1}(x_{T-1}, 1) \\
&+ \delta(1-\beta) P_{0,T-1}(x_{T-1}, 1) \sum_{x_T \in \mathcal{X}} V_T(x_T, 0) Q_0(x_T|x_{T-1})
\end{aligned} \tag{14}$$

For different action in the last period, the perceived long-run value functions are different because they will affect the continuation value.

Then they can be plugged into the choice-specific value function and CCP for period $T-2$ can be derived.

For the (paritally) naive agent, the problem is more complicated. First define the future self choice-specific value function:

$$z_{k_t}(x_t, k_{t-1}=0) = \begin{cases} u_0(x_t,0) + \tilde{\beta}\delta \sum\limits_{x_{t+1}\in\mathcal{X}} V_{t+1}(x_{t+1},0)Q_0(x_{t+1}|x_t), & k_t = 0 \\ u_1(x_t,0) + \tilde{\beta}\delta \sum\limits_{x_{t+1}\in\mathcal{X}} V_{t+1}(x_{t+1},1)Q_1(x_{t+1}|x_t), & k_t = 1 \\ u_2(x_t,0), & k_t = 2 \end{cases} \tag{15}$$

$$z_{k_t}(x_t, k_{t-1}=1) = \begin{cases} u_0(x_t,1) + \tilde{\beta}\delta \sum\limits_{x_{t+1}\in\mathcal{X}} V_{t+1}(x_{t+1},0)Q_0(x_{t+1}|x_t), & k_t = 0 \\ u_1(x_t,1), & k_t = 1 \\ u_2(x_t,1), & k_t = 2 \end{cases} \tag{16}$$

This is different from the current self choice-specific value function in the sense that it adopts the present-biased discount factor as $\tilde{\beta}$ rather than β. This is because at the current period, the partially naive agent thinks that she will use $\tilde{\beta}$ as present-biased discount factor rather than β in the future (while she will not behave like that when the future really comes). Also define the perceived CCP at $T-1$ period as

$$\tilde{P}_{k_t}(x_t, k_{t-1}) = \frac{exp(z_{k_t}(x_t, k_{t-1}))}{\sum\limits_{j\in\mathcal{D}} exp(z_j(x_t, k_{t-1}))} \tag{17}$$

Then the perceived long-run value function for period $T-1$ is

$$\begin{aligned} V_{T-1}(x_{T-1}, k_{T-2}=0) &= \mathbb{E}_{\epsilon_{T-1}}[u_{\tilde{\sigma}_{T-1}(x_{T-1},0,\epsilon_{T-1})}(x_{T-1},0) + \epsilon_{T-1}(\tilde{\sigma}_{T-1}) + \delta \sum_{x_T\in\mathcal{X}} V_T(x_T,\tilde{\sigma}_{T-1})Q_{\tilde{\sigma}_{T-1}}(x_T|x_{T-1})] \\ &= \mathbb{E}_{\epsilon_T}[z_{\tilde{\sigma}_{T-1}(x_{T-1},0,\epsilon_{T-1})}(x_{T-1},0) + \epsilon_{T-1}(\tilde{\sigma}_{T-1}) \\ &\quad + \delta(1-\tilde{\beta}) \sum_{x_T\in\mathcal{X}} V_T(x_T,\tilde{\sigma}_{T-1})Q_{\tilde{\sigma}_{T-1}}(x_T|x_{T-1})] \\ &= \gamma - \ln \tilde{P}_2(x_{T-1},0) + z_2(x_{T-1},0) \\ &\quad + \delta(1-\tilde{\beta})\tilde{P}_0(x_{T-1},0) \sum_{x_T\in\mathcal{X}} V_T(x_T,0)Q_0(x_T|x_{T-1}) \\ &\quad + \delta(1-\tilde{\beta})\tilde{P}_1(x_{T-1},0) \sum_{x_T\in\mathcal{X}} V_T(x_T,1)Q_1(x_T|x_{T-1}) \end{aligned} \tag{18}$$

$$
\begin{aligned}
V_{T-1}(x_{T-1}, k_{T-2}=1) &= \mathbb{E}_{\epsilon_{T-1}}[u_{\tilde{\sigma}_{T-1}(x_{T-1},1,\epsilon_{T-1})}(x_{T-1},1) + \epsilon_{T-1}(\tilde{\sigma}_{T-1}) + \delta \sum_{x_T \in \mathcal{X}} V_T(x_T, \tilde{\sigma}_{T-1}) Q_{\tilde{\sigma}_{T-1}}(x_T|x_{T-1})] \\
&= \mathbb{E}_{\epsilon_T}[z_{\tilde{\sigma}_{T-1}(x_{T-1},1,\epsilon_{T-1})}(x_{T-1},1) + \epsilon_{T-1}(\tilde{\sigma}_{T-1}) \\
&+ \delta(1-\tilde{\beta}) \sum_{x_T \in \mathcal{X}} V_T(x_T, \tilde{\sigma}_{T-1}) Q_{\tilde{\sigma}_{T-1}}(x_T|x_{T-1})] \\
&= \gamma - \ln \tilde{P}_2(x_{T-1},1) + z_2(x_{T-1},1) \\
&+ \delta(1-\tilde{\beta})\tilde{P}_0(x_{T-1},1) \sum_{x_T \in \mathcal{X}} V_T(x_T, 0) Q_0(x_T|x_{T-1})
\end{aligned}
\tag{19}
$$

Then they can be plugged into the choice-specific value function and CCP for period $T-2$ can be derived. For higher period case, it is easily extended from the three-period case.

3 Identification

The data record the state variable and the action decisions taken by the agent in each period. This section studies how the unknown primitives of the model, $Q_1, ..., Q_K; \beta; \delta; u_k(x)$ are identified. We focus on identifying the present-bias β and the discount factor δ. Given the observed state and decisions, the transition matrix Q_k can be directly identified.

Notice that the choice decision depend on the primitives through the value differences $w_k(x, k_{-1}) - w_K(x, k_{-1})$. Specifically, under the type-1 extreme value distribution, we have

$$
\ln\Big(\frac{p_k(x, k_{-1})}{p_K(x, k_{-1})}\Big) = w_k(x, k_{-1}) - w_K(x, k_{-1}), \tag{20}
$$

for all $k \in \mathcal{D}, x \in \mathcal{X}$. Thus, the choice specific value contrasts can be uniquely recovered from the observed choice probabilities. Thus, the question boils down to identify the present-bias and discount factors and flow utility functions using the identified choice specific value contrasts and transition matrix Q_k, which is pioneered in Hotz and Miller, 1993.

Before we examine the identification of the underlying model primitives, we first introduce the specific features in the reverse mortgage besides the general framework. In the reverse mortgage setting, we model the borrower in each period has possible options to choose from, i.e., $\mathcal{D} = \{0, 1, 2\}$, where choice 0 means that the agent chooses to continue (or make decision next period), and choice 2 means that the agent chooses to terminate the project immediately. For option 1, the project is terminated if the agent has already chosen option 1 in the previous period. Thus, choice 1 and choice 2 are both terminating actions. That is to say, $I_{k=2,k_{-1}=0}$, and $I_{k=1,k_{-1}=1} = 0$.

We then exploit the special feature of multiple terminating actions to rewrite the value of action

0 and 1 relative to the terminal action 2 as

$$w_k(x,k_{-1}) - w_2(x,k_{-1}) = \begin{cases} u_0(x,0) - u_2(x,0) + \beta\delta \sum_{x'} v(x',0)Q_0(x'|x), & k_{-1}=0, k=0 \\ u_0(x,1) - u_2(x,1) + \beta\delta \sum_{x'} v(x',0)Q_0(x'|x), & k_{-1}=1, k=0 \\ u_1(x,0) - u_2(x,0) + \beta\delta \sum_{x'} v(x',1)Q_1(x'|x), & k_{-1}=0, k=1 \\ u_1(x,1) - u_2(x,1), & k_{-1}=1, k=1 \end{cases} \tag{21}$$

Consequently, the log odd ratio of action 1 for different lagged action k_{-1} can be represented as

$$\begin{aligned} \Delta lnp_{12}(x) &\equiv \ln\left(\frac{p_1(x,0)}{p_2(x,0)}\right) - \ln\left(\frac{p_1(x,1)}{p_2(x,1)}\right) = [w_1(x,0) - w_2(x,0)] - [w_1(x,1) - w_2(x,1)] \\ &= [u_1(x,0) - u_2(x,0)] - [u_1(x,1) - u_2(x,1)] + \beta\delta \sum_{x'} v(x',1)Q_1(x'|x), \end{aligned} \tag{22}$$

which provides a moment condition regarding the product of the present-biased factor and the discount factor $\beta\delta$ and the long term value $v(x',1)$, while the log odd ratio $\Delta lnp_{12}(x)$ is identified from the data directly. This log odd ratio is determined by three components, the relative utilities $[u_1(x,0) - u_2(x,0)] - [u_1(x,1) - u_2(x,1)]$, the future discount factor $\beta\delta$, and the perceived long term value $v(x',1)$.

Due to the special features in the DDC models, that an individual makes decisions based on relative utilities instead of the absolute value of the utilities. Some restrictions regarding the utility functions are necessary.

Assumption 1 *The flow utility satisfies the following restriction.*

$$\begin{aligned} u_0(x,0) - u_0(x,1) &= h_0(x) \quad \textit{the punishment of missing payment in the previous year if continuing} \\ u_2(x,0) - u_2(x,1) &= 0 \quad \textit{the punishment of missing payment in the previous year if refinancing} \\ u_1(x,1) - u_1(x,0) &= u_2(x) \quad \textit{the extra cost due to termination by default} \end{aligned}$$

On the other hand, one can easily test on the exclusion restriction directly via checking how the previous action affecting the choice over continuing vs refinance. That is, by the terminate action feature, we have

$$\ln\left(\frac{p_0(x,0)}{p_2(x,0)}\right) - \ln\left(\frac{p_0(x,1)}{p_2(x,1)}\right) = [u_0(x,0) - u_2(x,0)] - [u_0(x,1) - u_2(x,1)] = h_0(x) = 0? \tag{23}$$

The component in the left-hand side can be directly identified and estimated from the data, while the component in the right-hand side equals to zero if the exclusion restriction is satisfied.

the following identification holds if the previous assumption is satisfied. If the restriction on the flow utility is satisfied, following the literature on identification in DDC models, we stack moment conditions in Equation 22 in the following matrix form:

$$\Delta ln\boldsymbol{p}_{12} = \boldsymbol{u}_2 + \beta\delta \boldsymbol{Q}_1 \boldsymbol{v}(1), \textit{depends on how we impose our restrictions on the flow utility.} \tag{24}$$

where $\Delta ln\boldsymbol{p}_{12} \equiv [\Delta lnp_{12}(x_1), ..., \Delta lnp_{12}(x_J)]'$ and $\boldsymbol{v}(k_{-1}) \equiv [v(x_1, k_{-1}), ..., v(x_J, k_{-1})]'$ are a $J \times 1$ column vector, and the transition matrix $\boldsymbol{Q}_1$ is a $J \times J$ matrix. From this moment condition, we can represent the perceived long run value directly as a function of the observed CCPs, the time-inconsistency, and the discount factor if the transition matrix $\boldsymbol{Q}_1$ is invertible.

To quantity the relationship between the long term value and the CCPs, we express the *perceived* long-run value function $v(x, k_{-1})$ following Equation 7. Note that the *perceived* long-run value function varies with the lagged action k_{-1}.

$$\begin{aligned} v(x,0) &= m(x,0) + u_2(x,0) + \delta(1-\beta)p_0(x,0)\sum_{x'\in\mathcal{X}} v(x',0)Q_0(x'|x) + \delta(1-\beta)p_1(x,0)\sum_{x'\in\mathcal{X}} v(x',1)Q_1(x'|x) \\ v(x,1) &= m(x,1) + u_2(x,1) + \delta(1-\beta)p_0(x,1)\sum_{x'\in\mathcal{X}} v(x',0)Q_0(x'|x), \end{aligned} \tag{25}$$

where $m(x, k_{-1}) \equiv \gamma - logp_K(x, k_{-1})$ is directly identified from the observed choice data.

If the restriction on the flow utility is satisfied, i.e., $u_2(x, 0) = u_2(x, 1)$, we then stack the above equations over all possible values of the state variable x and rewrite them in the following matrix form:

$$\begin{aligned} \boldsymbol{v}(0) &= \boldsymbol{m}(0) + \boldsymbol{u}_2 + \delta(1-\beta)\bar{\boldsymbol{Q}}_{00}\boldsymbol{v}(0) + \delta(1-\beta)\bar{\boldsymbol{Q}}_{10}\boldsymbol{v}(1) \\ \boldsymbol{v}(1) &= \boldsymbol{m}(1) + \boldsymbol{u}_2 + \delta(1-\beta)\bar{\boldsymbol{Q}}_{01}\boldsymbol{v}(0), \end{aligned} \tag{26}$$

where $\boldsymbol{u}_2$ is a $J \times 1$ vector of the flow utility of refinance, and $\bar{\boldsymbol{Q}}_{k,k_{-1}}$ is the mixture of the transition matrix and can be represented as

$$\bar{\boldsymbol{Q}}_{k,k_{-1}} \equiv \begin{bmatrix} p_k(x_1,k_{-1})\boldsymbol{Q}_k(x_1) \\ p_k(x_2,k_{-1})\boldsymbol{Q}_k(x_2) \\ \cdots \\ p_k(x_J,k_{-1})\boldsymbol{Q}_k(X) \end{bmatrix} = \begin{bmatrix} p_k(x_1,k_{-1})[Q_k(x_1|x_1), Q_k(x_2|x_1), \cdots, Q_k(x_J|x_1)] \\ p_k(x_2,k_{-1})[Q_k(x_1|x_2), Q_k(x_2|x_2), \cdots, Q_k(x_J|x_2)] \\ \cdots \\ p_k(x_J,k_{-1})[Q_k(x_1|x_J), Q_k(x_2|x_J), \cdots, Q_k(x_J|x_J)] \end{bmatrix},$$

which can be identified from the data directly. We then can cancel out the impact of the utility $\boldsymbol{u}_2$ on the perceived long-run value function, resulting in the following moment condition:

$$\boldsymbol{v}(0) - \boldsymbol{v}(1) = \boldsymbol{m}(0) - \boldsymbol{m}(1) + \delta(1-\beta)(\bar{\boldsymbol{Q}}_{00} - \bar{\boldsymbol{Q}}_{01})\boldsymbol{v}(0) + \delta(1-\beta)\bar{\boldsymbol{Q}}_{10}\boldsymbol{v}(1) \tag{27}$$

Our identification then proceeds in two steps. First of all, we show that the utility functions can be identified given the discount factor β and the time-inconsistency δ, which requires the following two rank conditions. Second, once the utility function being identified, the model generates polynomial moment conditions with the discount factor and the time-inconsistency as the only unknowns, allowing identification of both factors.

Assumption 2 $\boldsymbol{Q}_1$ *is of full rank.*

Assumption 3 $I - \delta(1-\beta)(\bar{\boldsymbol{Q}}_{00} - \bar{\boldsymbol{Q}}_{01})$ *is full rank*

With the full rank condition in Assumption 2, we can directly represent the value function $\boldsymbol{v}(1)$ as a function of the transition matrix and the two discount factors from Equation 24:

$$\beta^{-1}\delta^{-1}\boldsymbol{Q}_1^{-1}(-\boldsymbol{u}_2 + \Delta ln\boldsymbol{p}_{12}) = \boldsymbol{v}(1), \tag{28}$$

Next, we can express $\boldsymbol{v}(0)$ as a closed-form representation of the CCPs, discount factor β, δ and the identified $\boldsymbol{v}(1)$, so that it is identified if the discount factors are known from Equation 27. In addition, the flow utility $\boldsymbol{u_2}$ is identified from Equation 26 given the discount factor β, δ. Then flow utility function associated with other actions can be identified by Equation 20. We summarize this result in the following theorem.

Proposition 1 *Under Assumption 2 and 3, flow utility function $u_k(x, k_{-1})$ can be identified given the discount factor β and δ.*

We have exhaust all variations in the above theorem. Equations 24 and 26 just identifies $\boldsymbol{u}_2$, $\boldsymbol{v}(1)$ and $\boldsymbol{v}(0)$. Equation 21 just identifies the flow utility for other actions. To identify the discount factor β and δ, we need either normalization of utility or some exclusion restrictions.

If the flow utility function $u_2(x)$ are normalized for two states x_1 , x_2, then the discount factor can be identified. This is because given 26 and 24, we have

$$\frac{1}{\beta\delta}\frac{1}{\delta(1-\beta)}\bar{\boldsymbol{Q}}_{01}^{-1}\boldsymbol{Q}_1^{-1}\boldsymbol{g} - \frac{1}{\delta(1-\beta)}\bar{\boldsymbol{Q}}_{01}^{-1}(\boldsymbol{m}(1) + \boldsymbol{u}_2) - \frac{1}{\beta\delta}\bar{\boldsymbol{Q}}_{00}\bar{\boldsymbol{Q}}_{01}^{-1}\boldsymbol{Q}_1^{-1}\boldsymbol{g} + \bar{\boldsymbol{Q}}_{00}\bar{\boldsymbol{Q}}_{01}(\boldsymbol{m}(1) + \boldsymbol{u}_2) = \boldsymbol{m}(0) + \boldsymbol{u}_2 + \frac{\delta(1-\beta)}{\beta\delta}\bar{\boldsymbol{Q}}_{10}\boldsymbol{Q}_1^{-1}\boldsymbol{g} \tag{29}$$

where $\boldsymbol{g} = -\boldsymbol{u}_2 + \Delta ln\boldsymbol{p}_{12}$ Here we also assume that $\bar{\boldsymbol{Q}}_{01}$ is invertible. The expression 29 contains X equations and $X + 2$ unknown variables $\boldsymbol{u}_2$,β and δ. If the flow utility $-\boldsymbol{u}_2$ for at least two states are normalized, then the discount factor β and δ can be identified given some proper rank conditions.

3.1 identification of finite horizon

Only analyze the sophisticated agent case.

Although more and more panel data are available, there are still plenty of cases where it is hard to get the final period data, especially for the long term behavior like mortgage data. On the one hand, the time period is long (usually fifteen to thirteen years in mortgage market) so it would be hard to find the data for the last periods, on the other hand, there would be selective problems because agents could leave the project due to death or terminating. In this section, we would analyze the identification of discounting factors in finite horizon when the final period data are not available.

To make the question simplier, we assume that the reference utility function is invariant to time period, i.e. $u_{K,t}(x_t, k_{t-1}) = u_K(x_t, k_{t-1})$. Here the reference is action 2, so the reference utility function can be expressed as $u_2(x_t, k_{t-1})$. Similar to the infinite horizon case equation 22, we have the condition:

$$\begin{aligned}
\Delta lnp_{t12}(x_t) &\equiv \ln\left(\frac{p_{1,t}(x_t,0)}{p_{2,t}(x_t,0)}\right) - \ln\left(\frac{p_{1,t}(x_t,1)}{p_{2,t}(x_t,1)}\right) \\
&= [w_{1,t}(x_t,0) - w_{2,t}(x_t,0)] - [w_{1,t}(x_t,1) - w_{2,t}(x_t,1)] \\
&= [u_{1,t}(x,0) - u_{2,t}(x_t,0)] - [u_{1,t}(x_t,1) - u_{2,t}(x_t,1)] \\
&+ \beta\delta \sum_{x_{t+1}} v_{t+1}(x_{t+1},1) Q_1(x_{t+1}|x_t),
\end{aligned} \tag{30}$$

Here I make the similar assumption as infinite horizon case

Assumption 4 *The flow utility satisfies the following restriction.*

$u_{0,t}(x_t,0) - u_{0,t}(x_t,1) = h_t(x_t)$	*the punishment of missing payment in the previous year if continuing*
$u_2(x_t,0) - u_2(x_t,1) = 0$	*the punishment of missing payment in the previous year if refinancing*
$u_{1,t}(x_t,1) - u_{1,t}(x_t,0) = u_2(x_t)$	*the extra cost due to termination by default*

Under assumption 4, equation 30 can be rewritten (stacking over x_t) as

$$\Delta ln\boldsymbol{p}_{t12} = -\boldsymbol{u}_2 + \beta\delta \boldsymbol{Q}_1 \boldsymbol{v}_{t+1}(1), \textit{depends on how we impose our restrictions on the flow utility.} \tag{31}$$

Under assumption 3, v_{t+1} can still be expressed as function of discounting factor and the reference utility:

$$\beta^{-1}\delta^{-1}\boldsymbol{Q}_1^{-1}(\boldsymbol{u}_2 + \Delta ln\boldsymbol{p}_{t12}) = \boldsymbol{v}_{t+1}(1) \tag{32}$$

The other key identification condition is similar to the expression of perceived long-run value function 25, but in finite horizon case:

$$\begin{aligned}
v_t(x_t,0) &= m_t(x_t,0) + u_2(x_t,0) + \delta(1-\beta)p_{0,t}(x,0) \sum_{x_{t+1}\in\mathcal{X}} v_{t+1}(x_{t+1},0)Q_0(x_{t+1}|x_t) \\
&+ \delta(1-\beta)p_{1,t}(x_t,0) \sum_{x_{t+1}\in\mathcal{X}} v_{t+1}(x_{t+1},1)Q_1(x_{t+1}|x_t) \\
v_t(x_t,1) &= m_t(x_t,1) + u_2(x_t,1) + \delta(1-\beta)p_{0,t}(x_t,1) \sum_{x_{t+1}\in\mathcal{X}} v_{t+1}(x_{t+1},0)Q_0(x_{t+1}|x_t).
\end{aligned} \tag{33}$$

Stacking over x_t, the perceived long-run value function can be written as

$$\begin{aligned}
\boldsymbol{v}_t(0) &= \boldsymbol{m}_t(0) + \boldsymbol{u}_2 + \delta(1-\beta)\bar{\boldsymbol{Q}}_{00}\boldsymbol{v}_{t+1}(0) + \delta(1-\beta)\bar{\boldsymbol{Q}}_{10}\boldsymbol{v}_{t+1}(1) \\
\boldsymbol{v}_t(1) &= \boldsymbol{m}_t(1) + \boldsymbol{u}_2 + \delta(1-\beta)\bar{\boldsymbol{Q}}_{01}\boldsymbol{v}_{t+1}(0),
\end{aligned} \tag{34}$$

Assumption 5 $\bar{\boldsymbol{Q}}_{00}$,$\bar{\boldsymbol{Q}}_{01}$ *are of full rank*

If assumption 5 holds, $\boldsymbol{v}_{t+1}(0)$ can be cancelled out. Then $\boldsymbol{v}_t(0)$ can be expressed as

$$\boldsymbol{v}_t(0) \quad = \quad \boldsymbol{m}_t(0) + \boldsymbol{u}_2 + \delta(1-\beta)\bar{\boldsymbol{Q}}_{10}\boldsymbol{v}_{t+1}(1) + \bar{\boldsymbol{Q}}_{00}\bar{\boldsymbol{Q}}_{01}^{-1}(\boldsymbol{v}_t(1) - \boldsymbol{m}_t(1) + \boldsymbol{u}_2) \tag{35}$$

Since equation 31 holds for any time t, so $\boldsymbol{v}_t(0)$ can be expressed by β,δ and the reference utility, too. Then plug the expression of $\boldsymbol{v}_t(0)$, $\boldsymbol{v}_t(1)$ into the second equation of 34. The moment condition also contains X equations and $X+2$ unknowns (β,δ and $\boldsymbol{u}_2$). If two states of $u_2(x_t)$ are normalized, then β, δ and the rest of $u_2(x_t)$ can be identified. Then further $\boldsymbol{v}_t(0)$, $\boldsymbol{v}_t(1)$ and other flow utility can be identified. The difference between finite horizon and infinite horizon is that more non-stationary flow utility in $u_{0,t}(x_t, k_{t-1})$ and $u_{1,t}(x_t, k_{t-1})$ are allowed.

4 Monte Carlo Simulations

This is a simulation of borrower behavior in a reverse mortgage. In each period, the borrower can choose to continue, default, or terminate the loan. Especially, default can last for two consecutive periods, i.e. only when the borrower defaults for two periods, then the loan goes foreclosure. The borrower can also terminate the loan when she moves or repays the loan.

1. action set: $\mathcal{D} = \{0, 1, 2\}$. Action 0 means continuation, and action 1 means default, action 2 means termination. Specially, the loan will terminate if the agent chooses default for two consecutive periods. In this case, the agent's flow utility will be the utility from the loan plus the termination utility since the loan comes to an end.

2. state variable: $Supp(\mathcal{X}) = \{1, 2, 3, 4, 5, 6\}$, stands for credit left.

3. time horizon: in this simulation, I use $T = 3$, and draw $N = 30000$ agents.

4. discounting factor: the exponential discount factor $\delta = 0.8$ and the sophisticated present-biased discounting factor $\beta = 0.5$.

5. flow utility: following the exclusion restriction assumption 1:

$$u_k(x, k_{-1}) = \begin{cases} \theta_1 + \theta_2 x & k = 0, k_{-1} = 0 \\ \theta_3 + \theta_4 x & k = 0, k_{-1} = 1 \\ \theta_5 + \theta_6 x & k = 1, k_{-1} = 0 \\ (\theta_5 + \theta_7) + (\theta_6 + \theta_8)x & k = 1, k_{-1} = 1 \\ \theta_7 + \theta_8 x & k = 2 \end{cases} \tag{36}$$

   In the simulation, I set $\theta_1 = 2, \theta_2 = 1, \theta_3 = 3, \theta_4 = 1.5, \theta_5 = 1.3, \theta_6 = -1, \theta_7 = 3.3, \theta_8 = -1.2$. Following the normalization mentioned before, $u_2(x = 1) = u_2(x = 2) = 0$.

6. transition matrix: just for simplicity, set transition matrix as

$$Q_0(x'|x) = Q_1(x'|x) = \begin{bmatrix} 0.41 & 0.20 & 0.14 & 0.10 & 0.08 & 0.07 \\ 0.18 & 0.36 & 0.18 & 0.12 & 0.09 & 0.07 \\ 0.11 & 0.17 & 0.34 & 0.17 & 0.11 & 0.09 \\ 0.09 & 0.11 & 0.17 & 0.34 & 0.17 & 0.11 \\ 0.07 & 0.09 & 0.12 & 0.18 & 0.36 & 0.18 \\ 0.07 & 0.08 & 0.10 & 0.14 & 0.20 & 0.41 \end{bmatrix} \tag{37}$$

4.1 Estimation

I generate the actions of individual agents in the hyperbolic discounting framework. I then estimate the model primitives for both hyperbolic discounting and exponential discounting models. For comparison purpose, I present both estimates in Table 1

Table 1: estimation result of parameters

parameters	true value	present-biased model	s.e.	exponential model	s.e.
θ_1	2	2.01	(0.16)	1.96	(0.16)
θ_2	1	1.01	(0.11)	1.03	(0.11)
θ_3	3	3.06	(0.76)	2.81	(0.83)
θ_4	1.5	1.50	(0.35)	1.95	(0.39)
θ_5	1.3	1.30	(0.16)	1.33	(0.16)
θ_6	-1	-0.99	(0.12)	-1.01	(0.12)
θ_7	3.3	3.37	(0.69)	3.47	(0.74)
θ_8	-1.2	-1.22	(0.26)	-1.23	(0.27)
δ	0.8	0.80	(0.02)	0.42	(0.03)
β	0.5	0.50	(0.04)		

From Table 1, it can be seen that misspecification of exponential discounting model could underestimate δ since there is only one discounting factor, while the optimal model has two. The exponential discounting model also estimates some parameters in flow utility wrongly. For example, it overestimates θ_4. If the model is set with more periods, the effect from misspecification would be even larger. From the table, it temps to think that there is relation between present-biased model and exponential model in the sense that $\beta_p * \delta_p = \delta_e$ (here the subscript 'p' stands for present-biased discounting model and 'e' stands for the exponential discounting model). This is because the model simulated only consists of three periods. In this case the present-bias may not be severe becasue the agents only count two period ahead. If the model time horizon is longer, this pattern will disappear.

4.2 Counterfactual Analysis

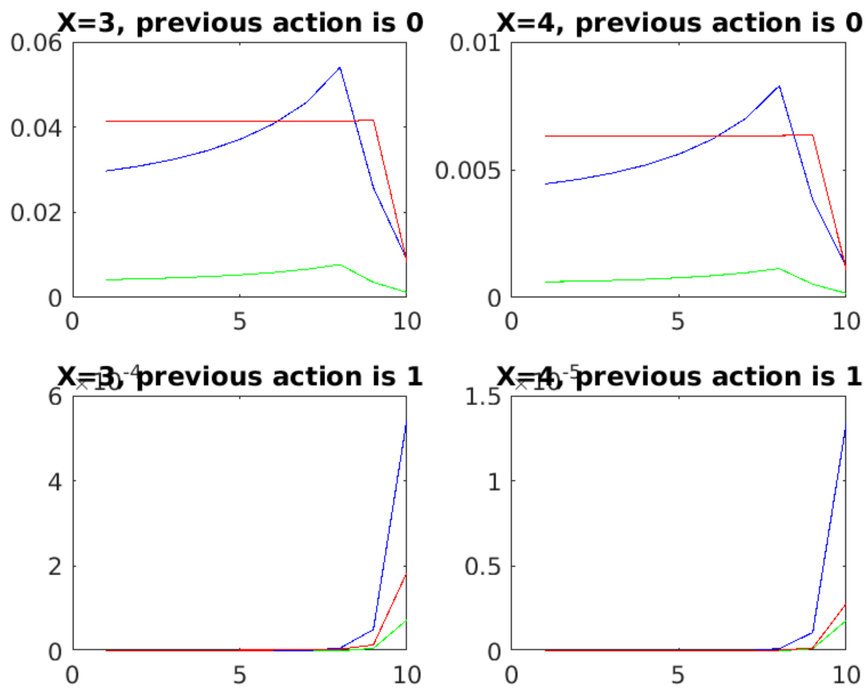
I explore the cases of intermediate punishment (or subsidy), and deferred punishment (or subsidy). To be more precise, I compare the cases when I give a subsidy $p = 2$ immediately when the agent chooses default, and the cases when I gives the subsidy $p(1+r)$ the period after the agent chooses default. To avoid the income effect, I set the interest rate $r = 1/(\beta * \delta) - 1 = 150\%$. The results are shown in figure 1 and figure 2. In figure 1, the green line shows the probability of choosing default rate under the present-biased model, which is the the benchmark case without any subsidy. The blue line shows the probability of choosing default rate under the present-biased model with intermediate subsidy, and the red line shows the probability of choosing default rate under the exponential model with intermediate subsidy. From the upleft panel of figure 1, when there is no subsidy to default, the default rate under the present-bias model first increases and then decreases. After the subsidy implies on default, the default rate does increase. However, the pattern is different for the exponential model. It shows that the default rate is stable at the beginning and then decreases a lot from the $T-2$ period. The reason is as following: by backward induction, in period T, the probability to default depends only on the flow utility because there is no future periods. This also explains why the default rates are the same for the red and blue line: under the same amount of intermediate subsidy, present-biased model and exponential model are the same in the final period. For period $T-1$, the choice specific value function consists of the flow utility and the continuation value. Since two model has different values estimated from the same data (see Table 1), the default rates are different. That is, if θ_1 to θ_8 are the same for two models and the $\beta\delta$ in present-biased model equals to the δ in exponential model, then the behavior in period $T-1$ will also be the same for two models.

The key difference between present-biased model and exponential model comes from period $T-2$: in this period, continuation value pattern is different for exponential model and present-biased model (see equation 13 and equation **??**, present-biased model has more complicated continuation value since β is not one, which is the exponential case). Since $w_2(x) = u_2(x)$, the introduction of extra term in continuation also changes the relative scale of choice specific value function, which makes the default rate even higher in this case. For periods before $T-2$, the patterns are similar to $T-2$. Depending on the values of θ, default rates are lower. Also, the introduction of the extra term also makes the default rate in present-biased model converge much more slowly than that in exponential model. This graphs shows that misspecification of model in discounting future could lead to big difference in counterfactual analysis. The upright panel of Figure 1 characterizes the case of difference x. In this model setting, higher x leads to lower default rate because θ_6 is set to be negative.

Different from upleft case, the downleft case shows monotone pattern. This is because when the agent defaults for two periods, the loan will terminate. Given the previous action is default already, choice specific value of default this period is not affected by the complicated continuation value, but only affected by teh flow utility (see equation 10). Then as the time far away from the final period, the default rate is much lower because it is more valuable to continue. Even with subsidy on default could not increase default rate much because the subsidy doesn't offset the big tempation of continuation.

Comparing different subsidy policy in Figure 1 and Figure 2, the default rate is higher in deferred subsidy for exponential model because of misspecification in discount factor.

Figure 1: default rate with intermediate subsidy

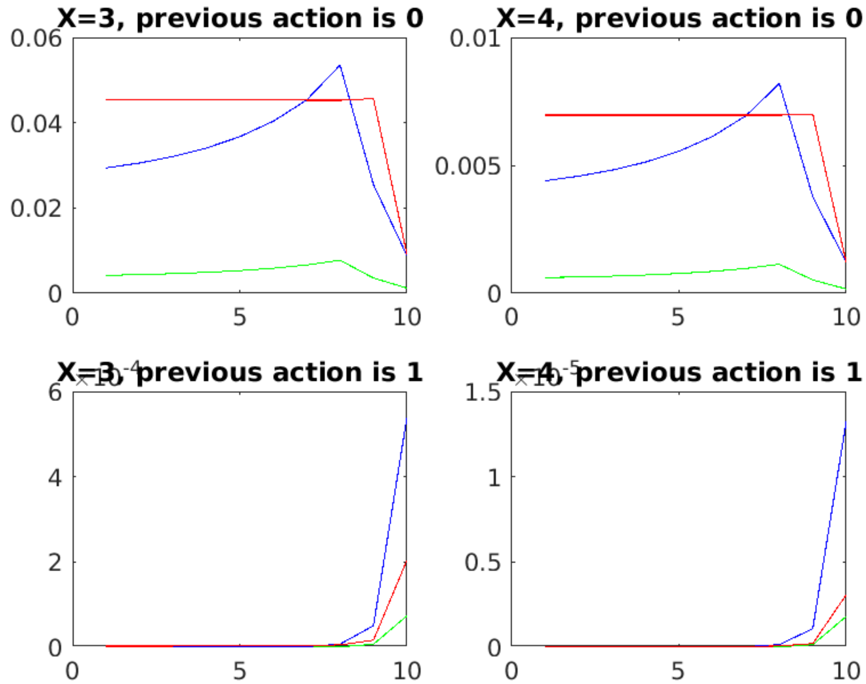


5 Conclusion

In this paper, I discuss the identificaiton of hyperbolic discounting factor in widely used dynamic discrete choice model with both finite horizon and infinite horizon. Under economically meaningful exclusion restrictions, the identification of disocunting factors is characterized by polynomial moment conditions. The presence of multiple terminating actions greatly reduces the complication of the identification and also helps relax the restrictions imposed on the flow utility function. In infinite horizon case, the joint identification of hyperbolic discounting factor and exponential discounting factor depends on the normalization of at least two states in the reference utility.

I also examine the effect of model misspecification by Monte Carlo simulations. When the underlying model is hyperbolic, estimation with exponential discounting model lead to wrong estimation in discounting factor and parameters in flow utility. I also find that mis-specification could result in misleading counterfactual outcomes.

Figure 2: default rate with deferred subsidy



References

[1] T. O'Donoghue and M. Rabin, "Doing it now or later," *American Economic Review*, no. March, pp. 103–124, 1999.

[2] H. Fang and Y. Wang, "Estimating dynamic discrete choice models with hyperbolic discounting, with an application to mammography decisions," *International Economic Review*, vol. 56, no. 2, pp. 565–596, may 2015. [Online]. Available: http://doi.wiley.com/10.1111/iere.12115

[3] J. H. Abbring and Ø. Daljord, "a Comment on "Estimating Dynamic Discrete Choice Models With Hyperbolic Discounting" By Hanming Fang and Yang Wang," *International Economic Review*, vol. 61, no. 2, pp. 565–571, may 2020. [Online]. Available: https://onlinelibrary.wiley.com/doi/abs/10.1111/iere.12434

[4] J. H. Abbring, O. Daljord, and F. Iskhakov, "Identifying Present-Biased Discount Functions in Dynamic Discrete Choice Models," *Working Paper*, 2019.

[5] A. Mahajan, C. Michel, and A. Tarozzi, "Identification of Time-Inconsistent Models: The Case of Insecticide Treated Nets," Tech. Rep., 2020. [Online]. Available: http://www.nber.org/papers/w27198

[6] M. K. Chan, “Welfare dependence and self-control: An empirical analysis,” *Review of Economic Studies*, vol. 84, no. 4, pp. 1379–1423, oct 2017. [Online]. Available: http://academic.oup.com/restud/article/84/4/1379/3091894/Welfare-Dependence-and-SelfControl-An-Empirical

[7] J. Rust, “Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher,” *Econometrica*, vol. 55, no. 5, p. 999, sep 1987.

[8] V. J. Hotz and R. A. Miller, “Conditional Choice Probabilities and the Estimation of Dynamic Models,” *The Review of Economic Studies*, vol. 60, no. 3, p. 497, jul 1993. [Online]. Available: https://academic.oup.com/restud/article-lookup/doi/10.2307/2298122

[9] J. R. Blevins, W. Shi, D. R. Haurin, and S. Moulton, “a Dynamic Discrete Choice Model of Reverse Mortgage Borrower Behavior,” *International Economic Review*, vol. 00, no. 00, 2020.

[10] A. Norets and X. Tang, “Semiparametric inference in dynamic binary choice models,” *Review of Economic Studies*, vol. 81, no. 3, pp. 1229–1262, 2014. [Online]. Available: https://academic.oup.com/restud/article-abstract/81/3/1229/1599945

[11] V. Aguirregabiria, J. Suzuki, V. Aguirregabiria, and J. Suzuki, “Identification and counterfactuals in dynamic models of market entry and exit and Wisconsin-Madison, for helpful comments and suggestions,” vol. 12, pp. 267–304, 2014.

[12] M. Kalouptsidi, P. Scott, and E. Souza-Rodrigues, “Identification of Counterfactuals in Dynamic Discrete Choice Models,” National Bureau of Economic Research, Cambridge, MA, Tech. Rep., sep 2015. [Online]. Available: http://www.nber.org/papers/w21527.pdf